# The Cube Root of Unity

Dan Stoneham

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$$z^3 = 1$$

$$z^3 - 1 = 0$$

$$(z-1)(z^2+z+1) = 0$$

So either

$$z - 1 = 0 \Rightarrow z = 1$$

or

$$z^2 + z + 1 = 0$$
$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$
$$z = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

For a slightly more advanced technique, we could also have solved using De Moivre's theorem, that is

$$z^3 = 1$$
$$z = 1^{\frac{1}{3}}$$
$$= (\cos(0) + i\sin(0))^{\frac{1}{3}}$$
$$= 1^{\frac{1}{3}} (\cos(0+2\pi n) + i\sin(0+2\pi n))^{\frac{1}{3}},\ n \in \mathbb{Z}$$
$$= \left(\cos\left(\frac{2\pi n}{3}\right) + i\sin\left(\frac{2\pi n}{3}\right)\right),\ n \in \mathbb{Z}$$

by De Moivre's Theorem, now letting $n = 0, 1, 2,$ we have,

$$z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

as we expect. Notice that putting in additional values of $n$ will give us repeat solutions. Indeed it is true that the three solutions are equally spaced about the unit circle on the Argand diagram.