Reduction Formula Examples

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Here we look at a number of reduction formula examples. We assume some basic working knowledge of reduction formula for integrals.

1. Let $I_n = \int_0^2 (4 - x^2)^n \, \mathrm{dx}$, show that $I_n = \frac{8n}{(2n+1)} I_{n-1}$, $n \geqslant 1$

We being this problem using integration by parts, letting dv be equal to 1, so we have

$$I_n = \int_0^2 (x)'(4 - x^2)^n \, \mathrm{dx}$$

$$= x(4 - x^2)^n|_0^2 - \int_0^2 n x (4 - x^2)^{n-1} \cdot (-2x) \, \mathrm{dx}$$

We notice that the first expression in this sum reduces to zero (since substituting in both 2 and 0 make part of the product zero), which leaves us to work with the integral. Rewriting this a bit, we are left with

$$I_n = 2n \int_0^2 x^2 (4 - x^2)^{n-1} \, \mathrm{dx}$$

The next part is probably the hardest part of the problem, although if you can spot it then it's pretty trivial. This is not the only way to solve this problem, but it's my preferred method. We notice that we can write I_n as

$$I_n = 2n \int_0^2 (-4 + 4 + x^2)(4 - x^2)^{n-1} \, \mathrm{dx}$$

$$= 2n \left[\int_0^2 (-4 + x^2)(4 - x^2)^{n-1} \, \mathrm{dx} + \int_0^2 4(4 - x^2)^{n-1} \, \mathrm{dx} \right]$$

$$= 2n[-I_n + 4I_{n-1}]$$

For the first integral, we pulled out a negative sign and multiplied the two remaining brackets together, to give us the original integral. The second integral is obviously $4I_{n-1}$. We are now ready to finish the problem

$$I_n = -2n I_n + 8n I_{n-1}$$

$$(2n+1) I_n = 8n I_{n-1}$$

$$I_n = \frac{8n I_{n-1}}{(2n+1)}, \; n \geqslant 1$$

as required. $\diamond$

2. Find a reduction formula for $I_n = \int \frac{\sin(nx)}{\sin x} \, \mathrm{dx}$, $n \geqslant 2$

We being this problem by writing $\sin(nx)$ as $\sin((n-1)x+x)$ and then expanding using the sum formula for sine. So we have

$$I_n = \int \frac{\sin((n-1)x+x)}{\sin x}\,\mathrm{dx}$$

$$= \int \frac{\sin[(n-1)x]\cos x + \cos[(n-1)x]\sin x}{\sin x}\,\mathrm{dx}$$

Remembering that the integral of a sum is the sum of the integrals, we write this as

$$= \int \frac{\sin[(n-1)x]\cos x}{\sin x}\,\mathrm{dx} + \int \cos[(n-1)x]\,\mathrm{dx}$$

We can evaluate the second expression immediately, the first one however requires a bit of clever thinking. We recall the lesser well known "product to sum" trigonometric identities, particularly

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

Applying this to the first integral we get

$$I_n = \int \frac{1}{2}\cdot\frac{\sin(nx) + \sin[(n-2)x]}{\sin x}\,\mathrm{dx} + \frac{1}{(n-1)}\sin[(n-1)x]$$

$$= \frac{1}{2}\int \frac{\sin(nx)}{\sin x} + \frac{\sin[(n-2)x]}{\sin x}\,\mathrm{dx} + \frac{\sin[(n-1)x]}{(n-1)}$$

As we can see, the end is in sight. We rewrite the above expression to get

$$2I_n = I_n + I_{n-2} + \frac{2\sin[(n-1)x]}{(n-1)}$$

$$I_n = \frac{2\sin[(n-1)x]}{(n-1)} + I_{n-2}, \quad n \geqslant 2$$

and we are done.

3. Find a reduction formula for $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\ \mathrm{dx}$

We begin with integration by parts, so

$$I_n = \int_0^{\frac{\pi}{2}} \left[\frac{x^{n+1}}{(n+1)}\right]' \sin x\ \mathrm{dx}$$

$$= \frac{x^{n+1}}{(n+1)}\sin x|_0^{\frac{\pi}{2}} - \frac{1}{(n+1)}\int_0^{\frac{\pi}{2}} x^{n+1}\cos x\ \mathrm{dx}$$

The first expression in the sum we can evaluate quite easily, the second expression is an integral with a cosine in it. Since our original integral had a sine in it, it might be a prusent course of action to integrate by parts for a second time. Indeed this is the route we shall take. So we have,

$$I_n = \frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)} - \frac{1}{(n+1)} \int_0^{\frac{\pi}{2}} \left[\frac{x^{n+2}}{(n+2)}\right]' \cos x \, \mathrm{dx}$$

$$= \frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)} - \frac{1}{(n+1)} \left[\frac{x^{n+2}}{(n+2)} \cos x \,|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{x^{n+2}}{(n+2)} \sin x \, \mathrm{dx} \right]$$

We we evaluate the first expression inside the square brackets we notice that this reduces to 0 - $0 = 0$. And so recognising that the second integral is simply $\frac{1}{(n+2)} I_{n+2}$, we have

$$(n+1) I_n = \left(\frac{\pi}{2}\right)^{n+1} - \frac{1}{(n+2)} I_{n+2}$$

At this point we may begin to wonder about our method of approach. We are trying to find a reduction formula, but the only thing we appear to have done here is increased n rather than reduced it. However we must remember that all we are trying to find is a relationship between some $\hat{n} \in \mathbb{N}$, $\hat{n} \geqslant 2$ and some $\hat{n} - a$, $a \in \mathbb{N}$. Well in this case we can let $\hat{n} = n + 2$, and so our reduction formula can now be written as,

$$\frac{1}{\hat{n}} I_{\hat{n}} = \left(\frac{\pi}{2}\right)^{\hat{n}\, -1} - (\hat{n} - 1) I_{\hat{n}\, -2}$$

$$\Rightarrow I_n = n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}, \;\; n \in \mathbb{N}, n \geqslant 2$$

and we are done.