Using Calculus to find the area of a circle, the surface area of a sphere and the volume of a sphere

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In this article we look at using a variety of integration skills to derive the well known results for the area of a circle, the surface area of a sphere and and volume of a sphere. We start by considering the unit circle centered at the origin. We make two changes, we restrict x and y to be non-negative, and we let the radius of the circle be of arbitrary length, which we shall call r. So we now have a function which looks something like this.

$$r^2 = x^2 + y^2,\ x \geq 0,\ y \geq 0$$

Note that this is now a quarter of a circle, that is the quarter in the first quadrant. We can now derive our results using this function. The area of a circle can be obtained by finding the area under this quarter circle and multiplying the result by four (to obtain the area of the whole circle). The area under the curve between 0 and r is given by

$$A = \int_0^r \sqrt{r^2 - x^2}\, dx$$

This looks like the kind of integral where a sine substitution would be helpful, and indeed it is. So we let $x = r \sin\theta$ which gives us

$$\frac{dx}{d\theta} = r\cos\theta \Rightarrow dx = r\cos\theta d\theta$$

We also need to consider the change in limits when we make this substitution. In this case the lower limit does not change since sine of zero is zero. But we also need $r = r\sin\theta$ and so this changes the upper limit to $\frac{\pi}{2}$ and we are now ready to write the integral after our substitution

$$A = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin\theta}\, r\cos\theta\, d\theta$$

$$= r^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\, \cos\theta\, d\theta$$

$$= r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

To find the integral of the square of cosine we remember the identity $\cos 2\theta = 2\cos^2\theta - 1$, which we can rearrange to give $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$, which we know how to integrate. So we now have

$$A = r^2 \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} \, d\theta$$

$$= r^2 \left[\frac{1}{4}\sin 2\theta + \frac{\theta}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\pi r^2$$

Since A was chosen to be a quarter of the circle it follows that the area of the circle is πr^2 as required.

We now turn our attention to spheres and try to find an expression for the surface area of a sphere. This is obtained by rotating the graph about the x-axis and taking the surface area generated, and doubling it to get the surface area of the whole sphere. I won't cover the derivation of the formula for surface area of revolution so you might want to look it up yourself. Nevertheless we arrive at the integral

$$S = 2\pi \int_0^r y(x) \left[\left(\frac{dy}{dx} \right)^2 + 1 \right]^{\frac{1}{2}} dx$$

We are required to find $\frac{dy}{dx}$ in this case (we may have considered parametric equations also if we wanted). We have that

$$y(x) = \sqrt{r^2 - x^2} \Rightarrow y'(x) = -x(r^2 - x^2)^{-\frac{1}{2}}$$

Squaring gives us

$$\left(\frac{dy}{dx} \right)^2 = \frac{x^2}{r^2 - x^2}$$

Our integral now looks a like

$$S = 2\pi \int_0^r \sqrt{r^2 - x^2} \left(\frac{x^2}{\sqrt{r^2 - x^2}} + 1 \right)^{\frac{1}{2}} dx$$

$$= 2\pi \int_0^r \sqrt{r^2 - x^2} \left(\frac{x^2}{r^2 - x^2} + 1 \right)^{\frac{1}{2}} dx$$

$$
\begin{aligned}
&= 2\pi \int_0^r \sqrt{r^2 - x^2} \left(\frac{x^2}{r^2 - x^2} + \frac{r^2 - x^2}{r^2 - x^2} \right)^{\frac{1}{2}} dx \\
&= 2\pi \int_0^r \sqrt{r^2 - x^2} \left(\frac{r}{\sqrt{r^2 - x^2}} \right) dx \\
&= 2\pi \int_0^r r \, dx \\
&= 2\pi \left[rx \right]_0^r \\
&= 2\pi r^2
\end{aligned}
$$

Since, as we discussed, this only accounts for half of our sphere. It follows that the surface area of the whole sphere is $4\pi r^2$ as required.

Finaly we look at the volume of the sphere. Not surprisingly, we rotate the graph in the first quadrant a full revolution and find the numerical value using the technique of finding the volume of revolution. Our volume for the whole sphere is simply double this result. So we have

$$
\begin{aligned}
V &= \int_0^r \pi y^2 \, dx \\
&= \pi \int_0^r r^2 - x^2 \, dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\
&= \pi \left[r^3 - \frac{1}{3} r^3 \right] = \frac{2}{3} \pi r^3
\end{aligned}
$$

However as discussed, this is only half of the volume, and so the total volume of the sphere is $\frac{4}{3}\pi r^3$ as required.